Lab 3. LAPLACE EQUATION $\nabla^2 = 0$

Name:------------------------- Date:----------------------------

1. Introduction

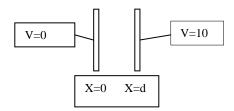The main goal of Lab 3 is to reproduce the results of Lab 2 using the MATLAB simulation tool on the computer.

We will use the PDE tool box to solve $\nabla^2 V = 0$, the equation for the potential $V(x, y, z)$ in a two dimensional region. We will be only using 3 of the two dimensional patterns used in Lab 2 (coaxial, 2 wire and the step)

MATLAB PDE Toolbox is available only to one student at a time in our lab. You can also use lab 109 Baskin Engineering, as well as remote access!

The " Getting Started " section of the partial differential equation tool box is quite confusing, because you have not solved partial differential equations before. ***Listen to the professor or TA's lecture at the start of the lab!! BE ON TIME.***

2. Discussion

For the square pattern given in lab 2 and shown below:

For this case V is just V(x), and so the continuous solution to

$\nabla^2 V = \dfrac{d^2 V}{dx^2} = 0$ with V(0)=0 and V(d)=10 is $V(x) = \dfrac{x}{d} V_o$. BUT the

problem is that the computer can NOT solve equations that are continuous in X, and so must be discretized. Discretize the interval $0 \le X \le d$ as $x = 0, x = \Delta, x = 2\Delta.........x = (n-1)\Delta, x = n\Delta = d$ NOTE that the discrete solution can be found analytically by solving $V(x) = V_0 x/d$ for

$x = 0, x = \Delta, x = 2\Delta.........x = (n-1)\Delta, x = n\Delta = d$ as the vector $\vec{V}$ where

$$\vec{V} = \begin{pmatrix} V(\Delta) \\ . \\ . \\ . \\ V((n-1)\Delta) \end{pmatrix} = \begin{pmatrix} \dfrac{\Delta}{d}V_0 \\ \dfrac{2\Delta}{d}V_0 \\ . \\ . \\ .\dfrac{(n-1)\Delta}{d}V_0 \end{pmatrix}$$

Then the equation for $\vec{V}$ can be found putting in the boundary conditions V(0)=0 and V(d)=10 as a separate vector, call it b, to make a matrix equation AV=b. (you figure out the matrix A and the vector b as explained below for the general case.)

GENERAL FORM:

Given the boundary value problem $\dfrac{d^2V(x)}{dx^2} = 0$, $V(0) = a, V(N\Delta) = b$,

Find $V(n\Delta)$ for n=1,2,........,N-1.

$$\dfrac{dV}{dx} \approx \dfrac{V(x+\Delta) - V(x)}{\Delta}$$

Answer: Discretize $\dfrac{d^2V(x)}{dx^2} = 0$ as $V((n+2)\Delta) - 2V((n+1)\Delta) + V(n\Delta) = 0$

$$\text{Let } \vec{V} = \begin{pmatrix} V(\Delta) \\ : \\ V(n\Delta) \\ : \\ V((N-1)\Delta) \end{pmatrix}$$

The following equations will be obtained:

$$V(2\Delta) = 2V(\Delta) - a$$
$$V(3\Delta) = 2V(2\Delta) - V(\Delta)$$
$$\vdots$$
$$\vdots$$
$$V((N-1)\Delta) = 2V((N-2)\Delta) - V((N-3)\Delta)$$
$$b = 2V((N-1)\Delta) - V((N-2)\Delta)$$

Writing the above equations in matrix form we obtain the following:

$$
\begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ b \end{pmatrix}
+
\begin{pmatrix}
0 & 1 & . & . & . & . \\
. & 0 & 1 & . & . & . \\
: & : & : & : & : & : \\
: & : & : & : & : & : \\
. & . & . & . & 0 & 1 \\
. & . & . & . & 0 & 0
\end{pmatrix}
\vec{V}
=
\begin{pmatrix}
2 & 0 & 0 & . & . & 0 \\
-1 & 2 & 0 & . & . & 0 \\
0 & -1 & 2 & . & . & 0 \\
: & : & : & : & : & : \\
: & : & : & : & : & : \\
: & : & : & 0 & -1 & 2
\end{pmatrix}
\vec{V}
+
\begin{pmatrix} -a \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}
$$

(1)　　　　(2)　　　　　　　　　　(3)　　　　　　　　(4)

Hint: call matrix (1)---- $\vec{b}$ and matrix (4)---- $\vec{a}$
　　　Call (N-1)×(N-1) matrix [D], where

$$
[D] =
\begin{pmatrix}
2 & -1 & 0 & . & . & . \\
-1 & 2 & -1 & . & . & . \\
0 & -1 & 2 & -1 & . & . \\
: & : & : & : & : & : \\
. & . & . & 0 & -1 & 0 \\
. & . & . & 0 & 2 & -1 \\
. & . & . & 0 & -1 & 2
\end{pmatrix}
$$

In the matrix form the following is obtained:
$$\vec{a} + \vec{b} = [D]\vec{V}$$

The solution is as follows:
$$\vec{V} = [D]^{-1}(\vec{a} + \vec{b})$$

Note: [D] has an inverse, (N-1)×(N-1) matrix, determinant ≠0.
Solution depends on $\vec{a}$ and $\vec{b}$.

3. Procedure

1. Set up each of the 3 patterns mentioned in the introduction section in lab 2 using the PDE toolbox, and solve them for V. Turn in the black/white copies of the colored solutions.

2. Export the V to show equipotential lines on the solutions. Mark the value of V on each equipotential line for 2.5, 5 and 2 Volts. Turn it in.

3. Express the $\vec{E} = -\nabla V$ and use the plot routine of MATLAB PDE toolbox to print arrows on your solution. Draw the field lines through the arrows to obtain field lines that are orthogonal to the equipotential lines. Turn it in.

4. Is the 5 volts equipotential line on pattern 2 a circle, and is it right in the middle between the inner and outer electrodes?

5. **_Compare_** and turn a discussion of the differences in Lab 2 and Lab 3. the differences should be small. But what would make the differences, and how small should they be?

6. Mathematically solve the total differential equation $\dfrac{d^2V(x)}{dx^2} = 0$ with boundary conditions at $V(x) = c_1$ at $x = 0$ and $V(x) = c_2$ at $x = 1$.

7. Why the potential V (x) does does not depend on the resistivity σ?

8. Discretize the ∇^2 operator on $\nabla^2 V(x, y) = 0$, similar to the discretization of V(x) in the discussion. **HOW DOES** that turn $\nabla^2 V(x, y) = 0$ into a vector/ matrix problem?
   NOTE that in the one dimensional case
   $\nabla^2 V \approx V((n+2)\Delta) - 2V((n+1)\Delta) + V(n\Delta)$ *which makes it*
   SYMMETRIC *about* $(n+1)\Delta$. *NOTE that in the 2- dimensional case the* ∇^2 *must be* **SYMMETRICAL** *in both x and y directions. Therefore, there is a central point from which the derivaties can be computed in each of the 4 directions, up, down, left and right.*

9. Put some reasonable numbers to compare the mathematical solution in 6 with the matrix solution of the discretized problem in the discussion.

10. Try to solve the discretized 2 dimensional problem in MATLAB for Fig 6.10 page 228 in the text book.

    Compare the 2 different methods:
    - MATLAB PDE toolbox
    - Discrete Method $\vec{V} = \left[D \right]^{-} \left(\vec{q} + \vec{b} \right)$
    - Fig given in the text book

11. Turn in a discussion of the similarities and differences of the total differential equation $\dfrac{d^2 V(x)}{dx^2} = 0$ in question 6 with the partial differential equation $\nabla^2 V(x, y) = 0$ with the boundary conditions prescribed at the edge of a 2 dimensional region as in question 8.

4. Things to turn in:

   a. Solutions of 5 patterns with equipotentials V=2.5, V=5, and 7.5 marked.
   b. Solutions of the 5 patterns with field lines drawn.
   c. Answer to question 4.
   d. Answer to question 5.
   e. Answer to question 6.
   f. Answer to question 7.
   g. The matrix solution to question 8 similar to the discretization of the total differential equation discussed in the discussion section.
   h. A discussion of the similarities and differences between solutions of total and partial differential equations.
   i. Answer to question 9.
   j. Answer to question 10, comparison of original figure in book (Fig 6.10), answers via solution of $\vec{V} = \left[D \right]^{-} \left(\vec{q} + \vec{b} \right)$, as well as answers from MATLAB PDE toolbox.